Graph Theory
Homework 3

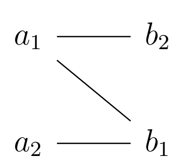
Joshua Ruiter

January 27, 2018

Exercise 1. Let $G = (V, E)$ be a bipartite graph, $V = A \sqcup B$. Let $\widetilde{G}$ be the network formed from G by vertices s, t, and directed edges $\{(s,a) : a \in A\} \cup \{(b,t) : b \in B\}$, and changing all edges in in G to directed edges in $\widetilde{G}$ pointing from A to B. We give each edge in $\widetilde{G}$ capacity 1. Finding a maximal flow in $\widetilde{G}$ is equivalent to finding a maximal matching in G.

Augmenting Path Algorithm	Maximal Matching Algorithm
1. Initialize a flow f_0 with value zero everywhere.	1. Define $M_0 = \emptyset$ to be our initial collection of edges in a matching.
2. Choose a below-capacity path from s to t and set $f_{k+1} = f_k + 1$ on each edge of that path.	2. Choose an edge $e \in E$ that doesn't use any vertex used by an edge in M_k, and set $M_{k+1} = M_k \cup \{e\}$.
3. Augment f_k using a backward path if possible.	3. If there is an edge $(a_1, b_1) \in M_k$ and edges $(a_1, b_2), (a_2, b_1) \in E$, with a_2, b_2 not already used by edges in M_k, then set $M_{k+1} = (M_k \setminus \{(a_1, b_1)\}) \cup \{(a_1, b_2), (a_2, b_1)\}$
4. Repeat steps 2-3 until no more augmenting is possible.	4. Repeat steps 2-3 until no more augmenting is possible.

The picture for step 3 of the Maximal Matching Algorithm looks like this. The diagonal edge is already in M_k, and the horizontal edges are the ones we replace it with.



Proposition 0.1 (Exercise 2ab)**.** *Let G be a finite group and H a subgroup. Then there is a set $\{g_1, \ldots, g_k\}$ of representatives for the left cosets of H that is simultaneously a set of representatives for the right cosets of H if and only if any collection of n distinct left cosets intersects at least n distinct right cosets.*

Proof. We form a bipartite graph $\widetilde{G}$ that contains information about cosets of G and apply the Hall Matching Theorem. The vertices of $\widetilde{G}$ are the collection of all left and right cosets of H in G. There is an edge between two cosets g_iH and Hx_j if $g_iH \cap Hx_j \neq \emptyset$.

$$V = L \sqcup R = \{g_1H, \dots, g_mH\} \sqcup \{Hx_1, \dots, Hx_m\}$$
$$E = \{(g_iH, Hx_j) : g_iH \cap Hx_j \neq \emptyset\}$$

Given a complete matching in $\widetilde{G}$ from L to R, we get simultaneous coset representatives, since each left coset has exactly one edge to a right coset, and this edge represents a simultaneous representative for those two cosets.

By Hall's Matching Theorem, a complete matching on this graph exists if and only if any set of n vertices in L has at least n neighbors in R. A collection of left cosets (vertices in L) has a neighbor for each right coset that intersects the union of that collection. That is, there is a complete matching if and only if for a collection of n distinct left cosets, their union intersects at least n distinct right cosets. □

Proposition 0.2 (Exercise 2c). *Let G be a group, and $H \subset G$ a finite subgroup. If $\{g_1H, \dots, g_nH\}$ is a collection of n distinct left cosets of H in G, then $\bigcup_i g_iH$ intersects at least n distinct right cosets of H in G.*

Proof. Let $\{Hx_1, \dots, Hx_k\}$ be the collection of (distinct) right cosets that intersect $\bigcup_i g_iH$. Since the right cosets cover all of G, this implies

$$\bigcup_{i=1}^{n} g_iH \subset \bigcup_{i=1}^{k} Hx_i$$

Since left cosets are pairwise disjoint, $\bigcup_i g_iH$ has $n|H|$ elements. Since right cosets are pairwise disjoint, $\bigcup_i Hx_i$ has $k|H|$ elements. Thus

$$\bigcup_{i=1}^{j} g_iH \subset \bigcup_{i=1}^{k} Hx_i \implies n|H| \leq k|H| \implies n \leq k$$

Thus $\bigcup_i g_iH$ intersects at least n disiinct right cosets. □

Interestingly, the previous proposition doesn't require G to be finite. However, the equivalence in Proposition 0.1 does need G to be finite, since Hall's Matching Theorem only includes finite graphs (as far as I know).

Corollary 0.3 (Exercise 2c). *If G is a finite group and H is a subgroup, there exists a set $\{g_1, \dots, g_k\}$ of representatives for the left cosets of H that is simultaneously a set of representatives for the right cosets of H.*

Proof. By Proposition 0.2, the second half of the equivalence in Proposition 0.1 always holds. □